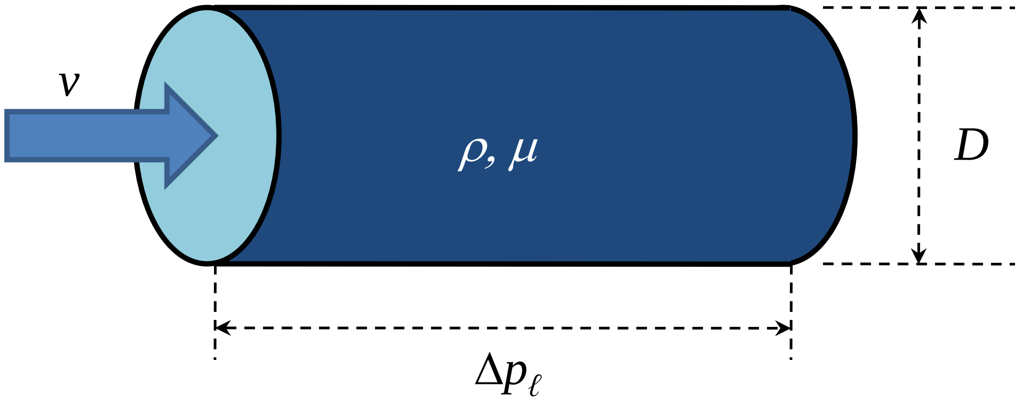
Facts

- A lot of fluid problems are too complicated to be solved by analytical procedures
- Needs a combination of theoretical, numerical, and experimental data
- Needs systematic methods to handle a bunch of variables in the experimental approach: *dimensional analysis*
- Needs a way to predict the behavior of real systems using models in the lab, the variables of which are under carefully controlled conditions: *concept of similitude*

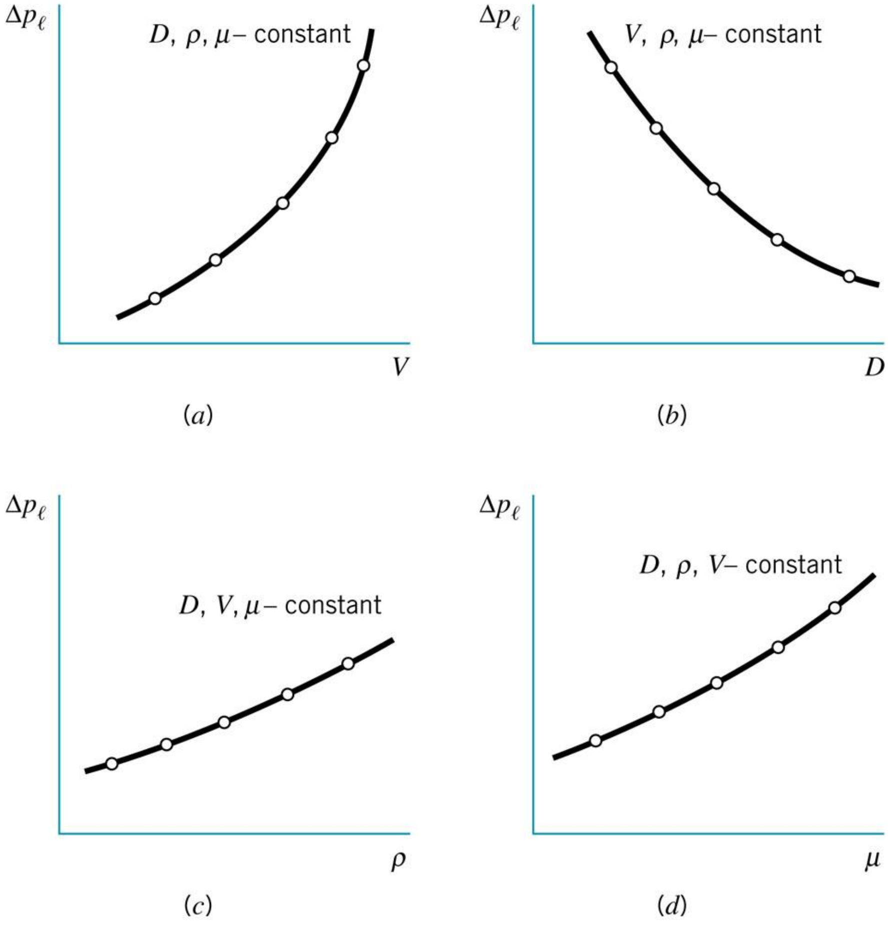
Dimensional Analysis

Suppose we want to establish the relationship among variables in a steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal, circular pipe.



$$\Delta p_{\ell} = f\left(D, \rho, \mu, v\right) \qquad \text{Eq (7.1)}$$

Dimensional Analysis

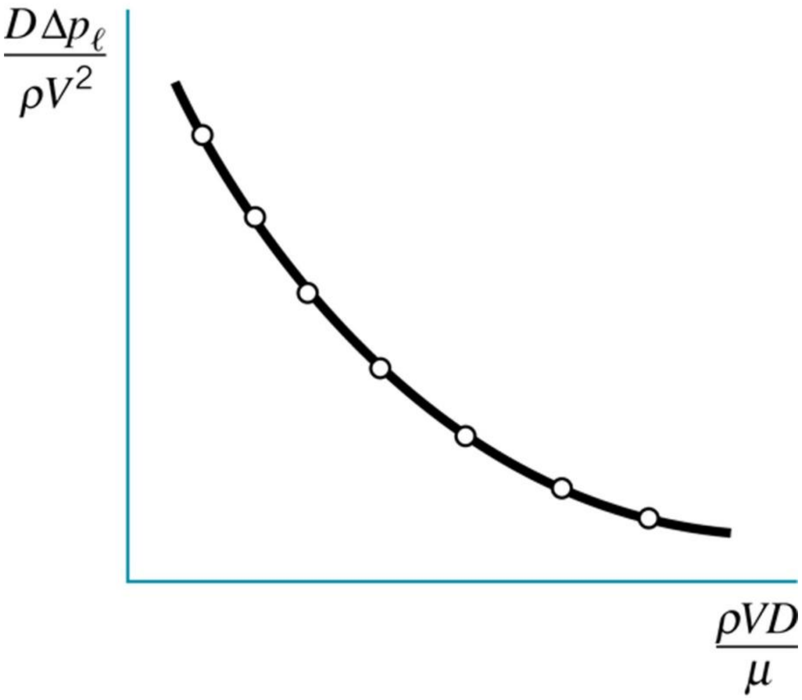


Dimensional Analysis

Instead of conducting experiments with varying all independent variables in turns like above, it is much simpler to group the variables into dimensionless products, such as here:

$$\frac{D\Delta p_\ell}{\rho v^2} = \phi\left(\frac{\rho v D}{\mu}\right)$$

Eq (7.2)



Buckingham Pi Theorem

- Dimensionless products are referred to as *pi terms*
- *k* variables that dimensionally homogeneous can be reduced to a relation among *k*–*r* independent *pi* terms (*r* is the minimum number of reference dimensions required to describe all variables)
- Selecting variables: 3 general groups (geometry, material properties, external effects)
- Generally we have the basic dimension: *FLT* or *MLT* as the reference dimension
- There is no unique set of *pi* terms, but the number of them are fixed by the theorem

Method of repeating variables

- List all variables that are involved in the problem (*k*)
- Express each of the variables in terms of basic dimensions (*r*)
- Determine the required number of *pi* terms: $k - r$
- Select a number of repeating variables (variables to appear in all *pi* terms): *r*
- Form the *p*i terms by multiplying each of the remaining $k - r$ variables with the product of the repeating variables (with each raised to some exponent to be determined such that the combination is dimensionless)
- Make sure that the resulting pi terms are indeed dimensionless
- Express the final form as a relationship among the *pi* terms:

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Common dimensionless groups

Reynolds number:

$$\mathrm{Re} = \frac{inertia}{viscous} = \frac{\rho v \ell}{\mu}$$

It is generally important in fluid dynamics. It is to determine whether a viscous flow is laminar or turbulent.

Mach number:

$$\mathrm{Ma} = \frac{inertia}{compressibility} = \frac{v}{c}$$

It is important in flows that account for the fluid compressibility.

Common dimensionless groups

Euler number:

$$\mathrm{Eu} = \frac{pressure}{inertia} = \frac{p}{\rho v^2}$$

It is used in problems in which pressure (pressure difference) are of interest.

Weber number:

$$\mathrm{We} = \frac{inertia}{surface\,tension} = \frac{\rho v^2 \ell}{\sigma}$$

It is important in flows that account for the fluid compressibility.

Common dimensionless groups

Froude number:

$$\mathrm{Fr} = \frac{inertia}{gravitational} = \frac{v}{\sqrt{g\ell}}$$

It is used in flow with a free surface such as river, lake, sea.

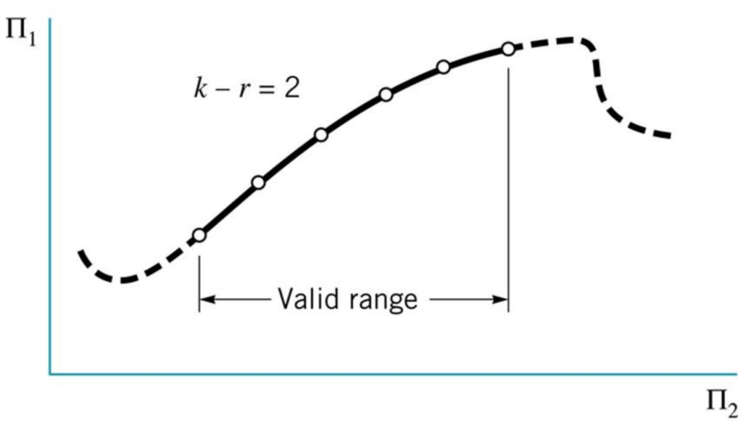
Strouhal number:

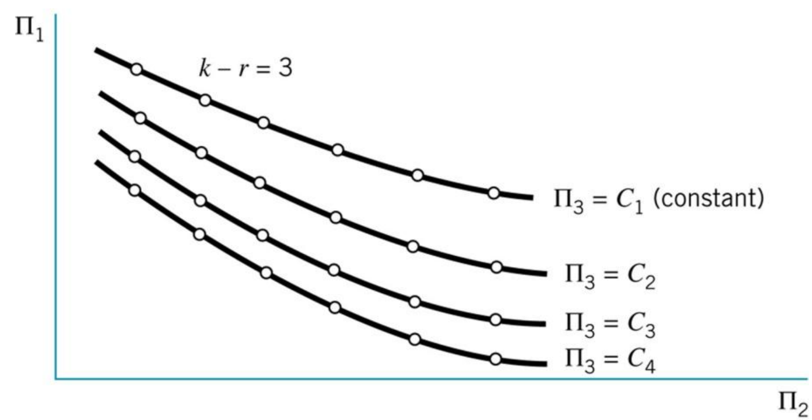
$$\mathrm{St} = \frac{local}{convective} = \frac{\omega\ell}{v}$$

It is used in unsteady flows with a characteristic frequency of oscillation.

Correlation of Experimental Data

- If there is only one pi term, then the relationship we are looking for is: $\prod_1$ = constant
- Graphical presentation of pi terms starts from two or more terms
- There is a valid range of the correlation, outside which extrapolation is dangerous





Modeling and Similitude

- A *model* is a representation of a physical system that may be used to predict the behavior of the system
- The physical model for which the prediction to be made is called the *prototype*
- There are equivalent different models: physical models, mathematical (analytical) models, and computer models
- Models are investigated in labs, thus have different size from that of the prototype (to be more easily handled and less expensive to construct and operate)
- Need to develop a proper procedure for designing models so that models and prototypes will behave in a similar fashion
- To have complete similarity between model and prototype, there must be geometric, dynamic, and kinematic similarity between the two systems

Modeling and Similitude

Model design conditions: similarity requirements

Prototype: $\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_n)$

Model: $\Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \ldots, \Pi_{nm})$

If the model is designed and operated under the following conditions:

$$\Pi_i = \Pi_{im} \qquad i = 2, \ldots, n \qquad \text{Eq (7.9)}$$

Then with a presumption that the form of ϕ is the same for model and prototype:

$$\Pi_1 = \Pi_{1m} \qquad \text{Eq (7.9)}$$

Modeling and Similitude

Scaling

$$\prod_i = \prod_{i\mathrm{m}}$$

for example: $$\frac{\ell}{D} = \frac{\ell_\mathrm{m}}{D_\mathrm{m}} \quad \rightarrow \quad \frac{\ell_\mathrm{m}}{\ell} = \frac{D_\mathrm{m}}{D}$$

Distorted model

One or more similarity requirements are not satisfied due to various reasons, e.g., a suitable fluid for the model cannot be found.

Distorted model can still be used, but with more difficult interpretation of the results.

Typical model studies

Flow through closed conduits

- No fluid interfaces and no free surface
- Dominant forces are inertial and viscous so that Reynolds number is an important similarity parameter. However, if Re is large then the viscous effect can be neglected.
- Liquids or gases can be considered incompressible as long as Mach number Ma < 0.3
- Be aware of the possibility of cavitation in the flow, which causes the vapor pressure to become an important variable. The *pi* term for this is the cavitation number: $(p_0 - p_v)/\rho v^2$.
- Geometric similarity must be maintained between model and prototype: $\prod_i = \ell_i/\ell = \prod_{im} = \ell_{im}/\ell_m$
- Also the surface roughness: ε/ℓ, particularly for turbulent flow

$$\prod_1 = \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \mathrm{Re}\right)$$

Typical model studies

Flow around immersed body

- No fluid interfaces and no gravitational effect
- Dominant forces are inertial and viscous so that Reynolds number is an important similarity parameter. However, if Re is large then the viscous effect can be neglected.
- Liquids or gases can be considered incompressible as long as Mach number Ma < 0.3
- Geometric similarity must be maintained between model and prototype: $\prod_i = \ell_i/\ell = \prod_{im} = \ell_{im}/\ell_m$
- Also the surface roughness: ε/ℓ, *p*articularly for turbulent flow
- The dependent variable of interest: drag and lift
- The *pi* term for drag is drag coefficient:

$$C_D = \frac{F_D}{\frac{1}{2}\rho v^2 \ell^2} = \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \mathrm{Re}\right)$$

Typical model studies

Flow with free surface

- There is a free surface with a liquid-air interface.
- Gravitational and inertial forces are important so that Froude number is an important similarity parameter.
- Forces due to surface tension may be significant so that Weber number becomes another similarity parameter.
- Geometric similarity must be maintained between model and prototype: $\prod_i = \ell_i/\ell = \prod_{im} = \ell_{im}/\ell_m$
- Also the surface roughness: ε/ℓ, particularly for turbulent flow
- Models involving free-surface flows are usually distorted.
- In many cases, surface tension and viscous effects are small, so that Froude number is the only parameter to consider.

$$\prod_1 = \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \mathrm{Re}, \mathrm{Fr}, \mathrm{Wb}\right) \approx \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \mathrm{Fr}\right)$$