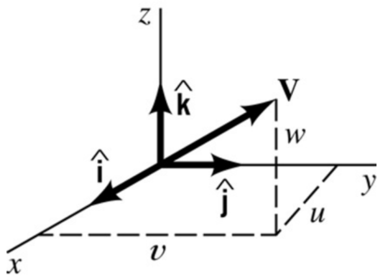
Differential Analysis

- The approach involves an infinitesimal control volume to produce details in fluid flow
- The governing differential equations for the flow of incompressible Newtonian fluids: **Navier – Stokes** (NSE) along with the continuity equation
- NSE can be generalized to include compressible fluids
- There are few known analytical solutions to NSE
- Numerical method is currently used: Computational Fluid Dynamics (CFD)

Differential Analysis

Fluid element kinematics

$$\mathbf{v} = v\hat{\mathbf{u}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$



Material derivative:

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \left(\frac{\partial \mathbf{v}}{\partial t} + v_x\frac{\partial \mathbf{v}}{\partial x} + v_y\frac{\partial \mathbf{v}}{\partial y} + v_z\frac{\partial \mathbf{v}}{\partial z}\right)$$

$$= \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{v}\right)$$

Eq (6.2-4)

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v}{\partial t}\hat{\mathbf{u}} + v\frac{\partial \hat{\mathbf{u}}}{\partial t}$$

Differential Analysis

Cartesian coordinates:

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$$

$$\nabla \equiv \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

Vector derivatives:

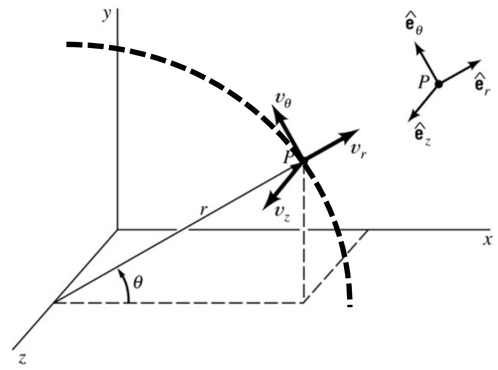
Gradient :

$$\nabla p \equiv \hat{\mathbf{i}} \frac{\partial p}{\partial x} + \hat{\mathbf{j}} \frac{\partial p}{\partial y} + \hat{\mathbf{k}} \frac{\partial p}{\partial z}$$

What are $\nabla x, \nabla y, \nabla z$?

Cylindrical polar coordinates:

$$\mathbf{v} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z$$ Eq (6.32)



$$\nabla p \equiv \hat{\mathbf{e}}_r \frac{\partial p}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial p}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial p}{\partial z}$$

Differential Analysis

Vector derivatives:

Divergence:

Volumetric dilatation rate:

$$\frac{1}{V}\frac{dV}{dt} = \nabla \bullet \mathbf{v}$$ Eq (6.9)

This rate causes a *linear* deformation.
For *steady* flow of *incompressible* fluid:

$$\nabla \bullet \mathbf{v} = 0$$ Eq (6.30)

Rotation/curl: vorticity

$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega}$$ Eq (6.17)

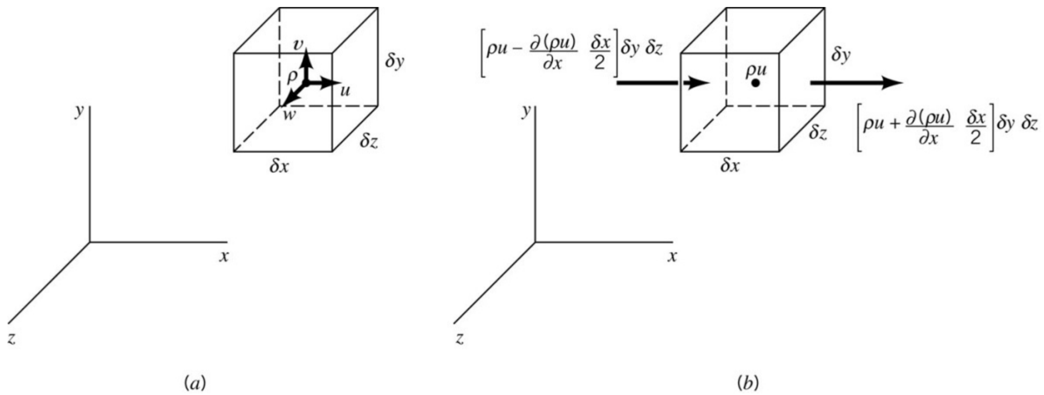
For *irrotational* flow field: $\nabla \times \mathbf{v} = \mathbf{0}$

Expressions in Cartesian and polar coordinates?

http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Differential Analysis

Conservation of mass



Integral form: $$\frac{\partial}{\partial t}\int_{cv}\rho dV + \int_{cs}\rho \mathbf{v}\bullet\hat{\mathbf{n}}dA = 0$$ Eq (6.19) Eq (5.5)

Differential form: $$\frac{\partial \rho}{\partial t} + \nabla\bullet\rho\mathbf{v} = 0$$ Eq (6.28)

Differential Analysis

Conservation of linear momentum

Integral form:

$$\frac{\partial}{\partial t}\int_{cv} \mathbf{v}\rho dV + \int_{cs} \mathbf{v}\rho\mathbf{v}\bullet\hat{\mathbf{n}}dA = \sum \mathbf{F}_{sys}$$

Eq (5.22)

Eq (6.44)

Differential form:

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \left(\mathbf{v}\bullet\nabla\right)\rho\mathbf{v} = \sum \mathbf{f}_{sys}$$

Eq (6.50)

For inviscid flow:

$$\sum \mathbf{f}_{sys} = \rho\mathbf{g} - \nabla p$$

Eq (6.52)

Euler's equation

For viscous flow:

$$\sum \mathbf{f}_{sys} = \rho\mathbf{g} + \nabla\overline{\overline{P}}$$

Eq (6.50)

Differential Analysis

$$\overline{\overline{P}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{zy} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

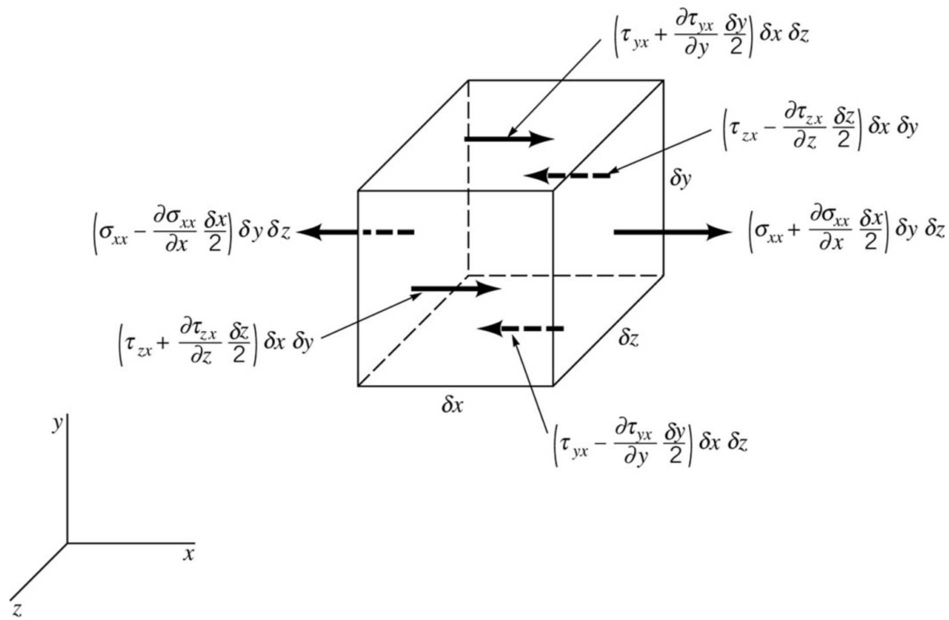
shear stresses

normal stresses

For inviscid flow:

$$\tau_{ij} = 0$$

$$\sigma_{ii} = -p$$



Bernoulli equation

For inviscid and incompressible flow:

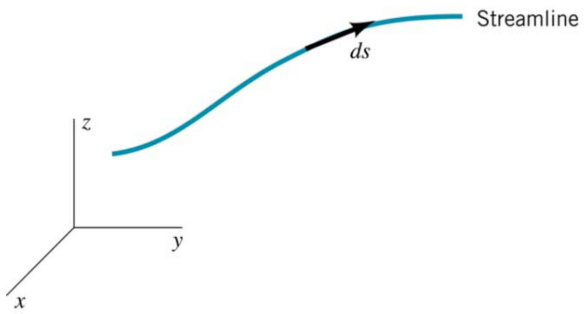
$$\rho(\mathbf{v}\bullet\nabla)\mathbf{v} = \rho\mathbf{g} - \nabla p$$

Eq (6.53)

Using vector identity: $(\mathbf{v}\bullet\nabla)\mathbf{v} = \tfrac{1}{2}\nabla(\mathbf{v}\bullet\mathbf{v}) - \mathbf{v}\times(\nabla\times\mathbf{v})$

$$\tfrac{1}{2}\rho\nabla(\mathbf{v}\bullet\mathbf{v}) - \rho\mathbf{v}\times(\nabla\times\mathbf{v}) = \rho g\nabla z - \nabla p$$

$$\frac{\nabla p}{\rho} + \tfrac{1}{2}\rho\nabla(v^2) + g\nabla z = \mathbf{v}\times(\nabla\times\mathbf{v})$$



Along a streamline:

$$\frac{\nabla p\bullet d\mathbf{s}}{\rho} + \tfrac{1}{2}\nabla(v^2)\bullet d\mathbf{s} + g\nabla z\bullet d\mathbf{s} = \mathbf{v}\times(\nabla\times\mathbf{v})\bullet d\mathbf{s}$$

Eq (6.54)

$$\frac{dp}{\rho} + \tfrac{1}{2}d(v^2) + g\,dz = 0$$

Eq (6.55)

Navier-Stokes equations

Stress-deformation relationships

For incompressible and Newtonian fluids:

Rate of angular deformation

$$\overline{\overline{P}} = \begin{bmatrix} -p + 2\mu \frac{\partial v_x}{\partial x} & \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \\ \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & -p + 2\mu \frac{\partial v_y}{\partial y} & \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) & \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) & -p + 2\mu \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Eq (6.125)

$$-p = \tfrac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

Rate of linear deformation

Navier-Stokes equations

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}\bullet\nabla)\mathbf{v}=\rho\mathbf{g}+\nabla\overline{\overline{P}}$$ Eq (6.50)

$$=\rho\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{v}$$ Eq (6.158)

In Cartesian coordinates:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\nabla^2 v_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\nabla^2 v_y$$ Eq (6.127)

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

where the *Laplacian*: $$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Navier-Stokes equations

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}\bullet\nabla)\mathbf{v} = \rho\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{v}$$ Eq (6.158)

In cylindrical polar coordinates: Eq (6.128)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\nabla^2 v_r-\frac{1}{r^2}\left(v_r+2\frac{\partial v_\theta}{\partial \theta}\right)$$

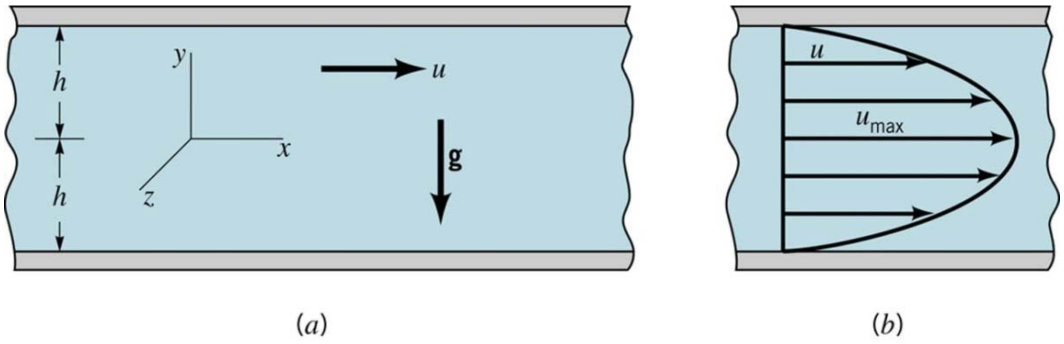
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\nabla^2 v_\theta-\frac{1}{r^2}\left(v_\theta-2\frac{\partial v_r}{\partial \theta}\right)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

where the *Laplacian*: $$\nabla^2\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

Simple solutions

Steady, laminar flow between fixed infinite parallel plates



$$v_y = v_z = 0$$ (one dimensional flow)

$$\frac{\partial v_x}{\partial x} = 0$$ (continuity equation: no change in speed)

$$\frac{\partial v_x}{\partial t} = 0$$ (steady flow)

$$v_x \equiv v_x(y)$$ (velocity profile)

Steady, laminar flow between fixed infinite parallel plates

Navier-Stokes equations:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\nabla^2 v_x$$

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v_x}{\partial y^2}$$ Eq (6.129)

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\nabla^2 v_y$$

$$0=-\rho g-\frac{\partial p}{\partial y}$$ Eq (6.130)

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

$$0=-\frac{\partial p}{\partial z}$$ Eq (6.131)

Steady, laminar flow between fixed infinite parallel plates

$$\left.\begin{aligned} 0 &= -\rho g - \frac{\partial p}{\partial y} \\ 0 &= -\frac{\partial p}{\partial z} \end{aligned}\right\} \rightarrow p = -\rho g y + f(x) \rightarrow \frac{\partial p}{\partial x} = f'(x)$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} \quad \rightarrow \quad v_x = \frac{f'(x)}{2\mu} y^2 + c_1 y + c_2$$ Eq (6.133)

Boundary conditions:

$v_x = 0$ at $y = \pm h$ (no-slip condition)

$$\rightarrow \quad v_x = \frac{f'(x)}{2\mu}\left(y^2 - h^2\right)$$ Eq (6.134)

Steady, laminar flow between fixed infinite parallel plates

Pressure gradient (along the flow):

$$\frac{\partial p}{\partial x} = f'(x) \rightarrow f'(x) = \frac{-\Delta p}{\ell}$$

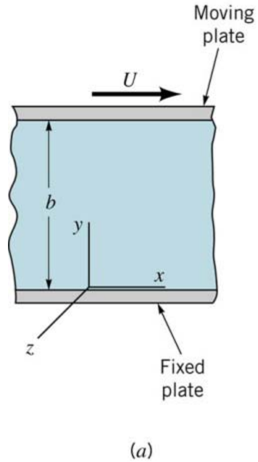
The volume flow rate per unit width (in *z* direction):

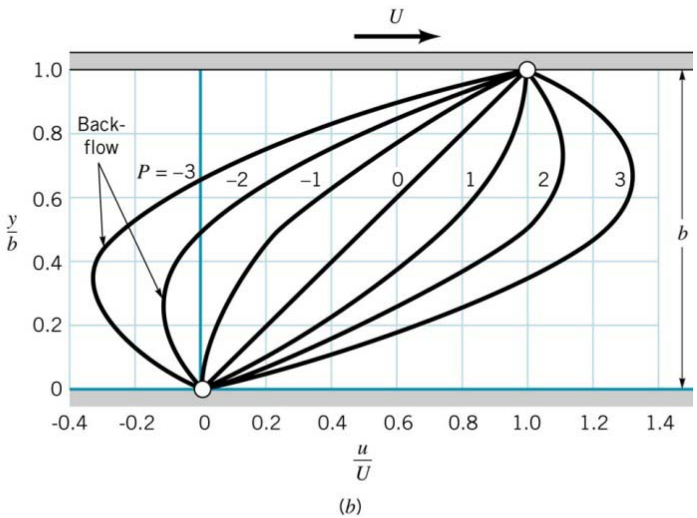
$$q = \int_{-h}^{h} v_x dy \quad = \frac{2}{3\mu}\frac{\Delta p}{\ell} h^3$$

Eq (6.136)

Simple solutions

Couette flow





$$v_x = \frac{f'(x)}{2\mu} y^2 + c_1 y + c_2$$

but now with boundary conditions:

$$v_x = 0 \quad \text{at } y = 0$$

$$v_x = U \quad \text{at } y = b$$

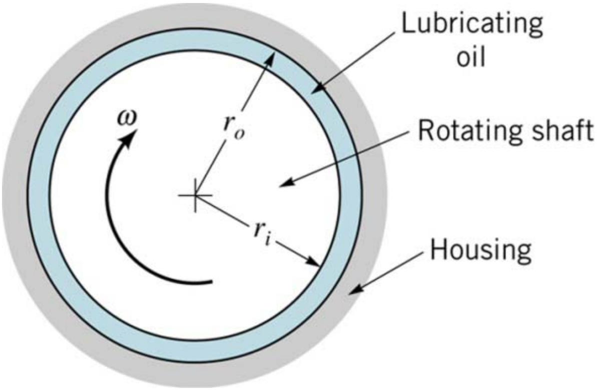
$$\rightarrow \quad v_x = \left(\frac{y}{b}\right)\left[U - \frac{b^2 f'(x)}{2\mu}\left(1 - \frac{y}{b}\right)\right]$$

Eq (6.140)

Couette flow

Special case:
pressure gradient $f'(x) = 0$

$$v_x = \left(\frac{y}{b}\right)U$$ Eq (6.142)



Flow in the narrow gap of a bearing:

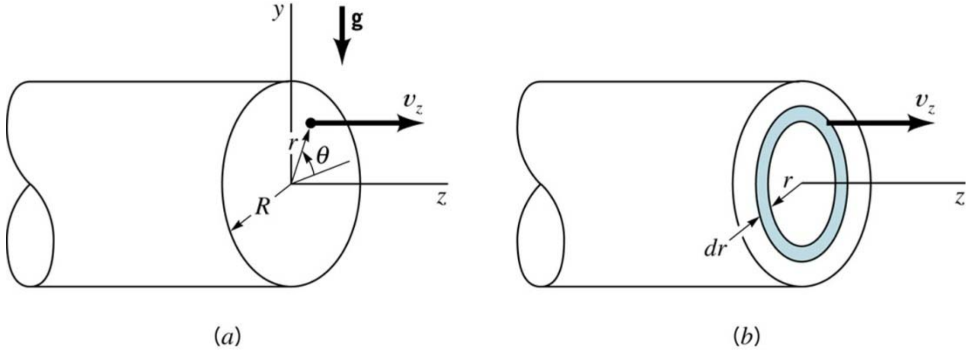
$$U = r_i\omega \quad \rightarrow \quad v_x = \frac{r}{r_o - r_i} r_i\omega$$

The shearing stress resisting the rotation:

$$\tau = \mu \frac{\partial v_x}{\partial r} \quad \rightarrow \quad \tau = \frac{\mu}{r_o - r_i} r_i\omega$$

Simple solutions

Steady, laminar flow in circular tubes



$$v_r = v_\theta = 0$$ (one dimensional flow, parallel to the wall)

$$\frac{\partial v_z}{\partial z} = 0$$ (continuity equation: no change in speed)

$$\frac{\partial v_z}{\partial t} = 0$$ (steady flow)

$$v_z \equiv v_z(r)$$ (velocity profile)

Steady, laminar flow in circular tubes

Navier-Stokes equations:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\nabla^2 v_r-\frac{1}{r^2}\left(v_r+2\frac{\partial v_\theta}{\partial \theta}\right)$$

$$0=-\rho g\sin\theta-\frac{\partial p}{\partial r}$$ Eq (6.143)

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\nabla^2 v_\theta-\frac{1}{r^2}\left(v_\theta-2\frac{\partial v_r}{\partial \theta}\right)$$

$$0=-\rho g\cos\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}$$ Eq (6.144)

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

$$0=-\frac{\partial p}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$ Eq (6.145)

Steady, laminar flow in circular tubes

$$\left.\begin{aligned} 0 &= -\rho g \sin\theta - \frac{\partial p}{\partial r} \\ 0 &= -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \end{aligned}\right\} \rightarrow p = -\rho g r \sin\theta + f(z) \rightarrow \frac{\partial p}{\partial z} = f'(z)$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad \rightarrow \quad v_z = \frac{f'(z)}{4\mu} r^2 + c_1 \ln r + c_2$$

Eq (6.147)

Boundary conditions:

$v_z = 0$ at $r = R$ (no-slip condition)

$v_z < \infty$ at $r = 0$

$$\rightarrow \quad v_z = \frac{f'(z)}{4\mu}\left(r^2 - R^2\right)$$

Eq (6.148)

Steady, laminar flow in circular tubes

Pressure gradient (along the flow):

$$\frac{\partial p}{\partial z} = f'(z) \rightarrow f'(z) = \frac{-\Delta p}{\ell}$$

The volume flow rate:

$$Q = \int_0^R v_z (2\pi r dr) \quad = \frac{\pi R^2}{8\mu}\frac{\Delta p}{\ell}$$

Eq (6.151)

Poiseuille's law